B.Sc.(Math) part-II

paper - III

Topic:- Fundamental theorem of homomorphism of groups.

Statement!- Every homomorphic image of group G is isomorphic to some quotient group of G.

proof:- Let f be a homomorphism of G onto G' then G' is the homomorphic image of group G

we want to show that G' is isomorphic to some quotient.

Let K be kernel of homomorphism f Then K is a normal subgroup of G we shall prove that $G/K \cong G'$

Let $a \in G$ then $f(a) \in G'$ since $f: G \to G'$

Also if $a \in G$ then $Ka \in G/K$ because G/K is the set of all right (left) cosets of K in G

Consider the mapping $\phi: G/K \to$ such that $\phi\{Ka\} = f(a)$ for all $a \in G$.

we shall show that ϕ is an isomorphism of G/K onto G

(i) ϕ is one-one we have

$\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$

$\Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$= f(a) f(b^{-1}) = e' = f(ab^{-1}) = e'$

$\because f$ is homomorphism

$\Rightarrow ab^{-1} \in K \because K$ is Kernal i.e

a belongs to coset Kb

$\Rightarrow Ka = Kb$

$\therefore \phi$ is one-one.

(ii) ϕ is onto

Let y be any element of G'

since f is onto G' there exists an element a such that $f(a) = y$

Now $Ka = G/K$ and we have

$\phi(Ka) = f(a) = y$.

which means that the preimage of an element $y \in G'$ is Ka under the mapping ϕ

Hence ϕ is onto

(iii) ϕ preserves operations

we have $\phi[(Ka)(Kb)]$

$= \phi(Kab) = f(ab)$

by the def of ϕ

$= f(a) f(b) \because f$ is homomorphism

$= \phi(Ka)\phi(Kb)$

$\therefore \phi$ is an isomorphism of G/K onto G'

Hence $G/K \cong G'$

Second form of Fundamental theorem:

Theorem:- Let G be a group and let H be a normal subgroup of G. Then the mapping $f: G \to G/H$ given by $f(x) = xH$ where $x \in G$ is an onto homomorphism. The Kernal of f is the normal subgroup H. In other words, any quotient group G/H of a group G w.r.t a normal subgroup H, is a homomorphic image of G.

proof:- Let $x, y \in G$ then

$$f(xy) = xyH = xH \cdot yH = f(x) f(y)$$

Which ~~show~~ means that f is a homomorphism.

Moreover, any element of G/H is of the form xH where $x \in G$ and under the mapping f, it is image of x

There f is an onto homomorphism

Now want to prove that Kernal of f is H.

For this we see that the image of all points $h \in H$ is $f(h) = h(H) = H$ which is the identity element of G/H. Hence that Kernal of f is H.

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